Eulerian representations for Coincidental reflection groups

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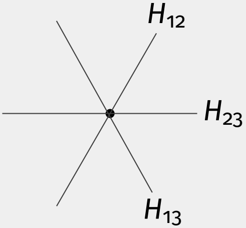

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Arrangements at Home III: Algebraic Aspects
August 13, 2020

Big Idea:

Generalize a beautiful Type *A* story connecting combinatorics, representation theory and topology to a broader class of reflection groups

Outline:

1. Motivating Story: Type A
2. Coincidental reflection groups
3. The Varchenko-Gelfand ring
4. Eulerian idempotents
5. Main Results

Motivating Story: Type A

Complement of the Braid Arrangement

Start with the real **Braid arrangement**

$$\mathcal{A} = \{H_{ij} = \{x_i - x_j = 0\} : 1 \leq i < j \leq n\} \subset \mathbb{R}^n$$

and **complement**

$$\mathcal{M}(\mathcal{A}) := \mathbb{R}^n \setminus \mathcal{A} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \neq x_j\}.$$

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We are interested in the ***d*-thickened complement**

$$\begin{aligned}\mathcal{M}^d(\mathcal{A}) :=& \mathcal{M}(\mathcal{A}) \otimes \mathbb{R}^d = \mathbb{R}^{dn} \setminus \Big(\bigcup_{1 \le i < j \le n} H_{ij} \otimes \mathbb{R}^d \Big) \\ =& \{(x_1, \cdots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j\} \\ =& \mathrm{Conf}_n(\mathbb{R}^d).\end{aligned}$$

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Example: When $d = 2$, this is equivalent to the complement of the complexified arrangement $\mathcal{M}(\mathcal{A}) \otimes \mathbb{C}$

A natural question:

$$\text{what is } H^*(\mathrm{Conf}_n(\mathbb{R}^d), \mathbb{R})?$$

Cohomology presentation

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Theorem (Arnol'd (1969): $d = 2$, F. Cohen (1976): $d \geq 2$).

The ring $H^* \mathrm{Conf}_n(\mathbb{R}^d)$ has presentation

$$\mathbb{R}\langle e_{ij} : 1 \leq i \neq j \leq n \rangle / \mathcal{J}$$

where each e_{ij} is in degree $d - 1$ and $\mathcal{J}$ is generated by

1. e_{ij}^2
2. $e_{ij} = (-1)^d e_{ji}$
3. $e_{ij}e_{j\ell} + e_{j\ell}e_{\ell i} + e_{\ell i}e_{ij}$

for any $1 \leq i \neq j \neq \ell \leq n$.

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This implies that $H^* \mathrm{Conf}_n(\mathbb{R}^d)$ is

concentrated in degrees $k(d - 1)$ for $0 \leq k \leq n - 1$

commutative when d is **odd**

anti-commutative when d is **even**

Representations?

The symmetric group S_n acts on

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What representation does $H^{k(d-1)} \,\mathrm{Conf}_n(\mathbb{R}^d)$ carry for each k?

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A more refined question:
What representation does $H^{k(d-1)} \mathrm{Conf}_n(\mathbb{R}^d)$ carry for each k?

When d is **odd**,
the answer is linked to classical objects in combinatorics
called the **Eulerian idempotents**...

Let $\mathbb{R} S_n$ be the group algebra of S_n and $w = (w_1, w_2, \cdots w_n) \in S_n$.

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Definition (Barr, 1968).

The **Shuffle (Barr) element** in $\mathbb{R}\,S_n$ is

$$\mathcal{S} := \sum_{i=1}^{n-1} \sum_{\substack{w \in S_n: \\ w_1 < \cdots < w_i \\ w_{i+1} < \cdots < w_n}} w \in \mathbb{R}\,S_n.$$

Example: When $n = 3$,

$$\mathcal{S} = \underbrace{(\mathbf{1}, \mathbf{2}, \mathbf{3}) + (\mathbf{2}, \mathbf{1}, \mathbf{3}) + (\mathbf{3}, \mathbf{1}, \mathbf{2})}_{i=1} + \underbrace{(\mathbf{1}, \mathbf{2}, \mathbf{3}) + (\mathbf{1}, \mathbf{3}, \mathbf{2}) + (\mathbf{2}, \mathbf{3}, \mathbf{1})}_{i=2}$$

$$= 2(1, 2, 3) + (2, 1, 3) + (3, 1, 2) + (1, 3, 2) + (2, 3, 1).$$

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Corollary.

By Lagrange interpolation, the idempotent projecting onto the σ_k-th eigenspace of $\mathcal{S}$ is

$$\mathfrak{e}_k := \prod_{j \neq k} \frac{\mathcal{S} - \sigma_j}{\sigma_k - \sigma_j}.$$

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Note:

By construction, $\mathbb{R}\,S_n \mathfrak{e}_k$ is the σ_k-eigenspace of $\mathcal{S}$

Example: When $n = 3$, the Barr element $\mathcal{S}$ has eigenvalues $0, 2, 6$:

$$\mathfrak{e}_0 = \frac{(\mathcal{S} - 2)(\mathcal{S} - 6)}{(0-2)(0-6)} \qquad \sigma_0 = 0\text{-eigenspace projector}$$

$$= \frac{1}{6}\big((1,2,3) - (2,1,3) - (3,1,2) - (1,3,2) - (2,3,1) + 2(3,2,1)\big)$$

$$\mathfrak{e}_1 = \frac{(\mathcal{S} - 0)(\mathcal{S} - 6)}{(2-0)(2-6)} \qquad \sigma_1 = 2\text{-eigenspace projector}$$

$$= \frac{1}{2}\big((1,2,3) - (3,2,1)\big)$$

$$\mathfrak{e}_2 = \frac{(\mathcal{S} - 0)(\mathcal{S} - 2)}{(6-0)(6-2)} \qquad \sigma_2 = 6\text{-eigenspace projector}$$

$$= \frac{1}{6}\big((1,2,3) + (2,1,3) + (3,1,2) + (1,3,2) + (2,3,1) + (3,2,1)\big)$$

Connections to Configuration spaces

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Key connection:

When $d \geq 3$ is **odd**, for $0 \leq k \leq n-1$,

$$H^{k(d-1)} \operatorname{Conf}_n(\mathbb{R}^d) \cong_{S_n} \mathbb{R} S_n \mathfrak{e}_{n-1-k}.$$

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1997 Sundaram-Welker prove an equivariant formulation of the Goresky-MacPherson formula relating

cohomology of a subspace arrangement $\longleftrightarrow$ homology of its intersection lattice

As a special case:

they compute the characters of $H^k\,\mathrm{Conf}_n(\mathbb{R}^d)$

Descents

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For $w = (w_1, \cdots, w_n) \in S_n$, the **descent set** of w is

$$\mathrm{Des}(w) := \{i \in [n-1] : w_i > w_{i+1}\}$$
$$= \Big\{ \underbrace{s_i}_{\substack{\text{transposition}\\ (i,i+1)}} \in \underbrace{S}_{\substack{\text{Coxeter}\\ \text{generators}}} : \underbrace{\ell(ws_i) < \ell(w)}_{\text{Coxeter length}} \Big\}.$$

The **descent number** of w is

$$\mathrm{des}(w) := \#\,\mathrm{Des}(w).$$

Example: If $w = (1, \mathbf{3}, 2, \mathbf{5}, 4)$, then

$$\mathrm{Des}(w) = \{\mathbf{2}, \mathbf{4}\} = \{s_{\mathbf{2}} = (23), s_{\mathbf{4}} = (45)\},$$
$$\mathrm{des}(w) = 2.$$

Remark: **Des** and **des** can be defined for any Coxeter group.

Surprising fact: (Solomon, 1976)

There is a *subalgebra* of $\mathbb{R}\, S_n$ generated by sums of elements with the same descent set:

$$\mathcal{D}(S_n) := \left\langle Y_T := \sum_{\substack{w \in S_n \\ \mathrm{Des}(w)=T}} c_T w : c_T \in \mathbb{R}, T \subset [n-1] \right\rangle$$

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Example: When $n = 3$, the descent algebra $\mathcal{D}(S_3)$ has basis:

$$\begin{aligned} Y_{\emptyset} &= (1,2,3) \\ Y_1 &= (2,1,3) + (3,1,2) \\ Y_2 &= (1,3,2) + (2,3,1) \\ Y_{1,2} &= (3,2,1). \end{aligned}$$

Remark: In fact, $\mathcal{D}(W)$ is a subalgebra for any Coxeter group.

Theorem (Garsia-Reutenaur, 1989).

The **Eulerian idempotents** are defined by the equation

$$\sum_{k=0}^{n-1} t^{k+1}\mathfrak{e}_k = \sum_{w\in S_n} \binom{t-1+n-\mathrm{des}(w)}{n} w.$$

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Eulerian idempotents, redefined

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The $\mathfrak{e}_k$ are in $\mathcal{D}(S_n)$ and generate a *commutative subalgebra* spanned by sums of elements with the same descent number

(This algebra is called the **Eulerian subalgebra)**

Example: When $n = 3$,

$$\mathfrak{e}_0 = \frac{1}{6}\big((1,2,3) - (2,1,3) - (3,1,2) - (1,3,2) - (2,3,1) + 2(3,2,1)\big)$$
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$$= \frac{1}{6}\big(Y_{\emptyset} + Y_1 + Y_2 + Y_{1,2}\big)$$

Summary: For $0 \leq k \leq n-1$, the following are equivalent as S_n-representations:

1. $H^{k(d-1)} \operatorname{Conf}_n(\mathbb{R}^d)$ for $d \geq 3$ and odd
2. The $\sigma_{n-1-k} = \{2^{n-k} - 2\}$-eigenspace of the shuffle operator $\mathcal{S}$
3. The representation $\mathbb{R} S_n \mathfrak{e}_{n-1-k}$, where $\mathfrak{e}_{n-1-k}$ is defined by

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Type A Summary

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Goal:

Generalize this statement to coincidental reflection groups,
i.e. reflection groups whose exponents form an arithmetic progression

$$1, 1+g, 1+2g, 1+3g, \cdots$$

Coincidental Analog

Recall the rising factorial $(t)_k := (t)(t+1)\dots(t+k-1)$ and let

$$\beta_{W,k}(t) := \frac{\left(\frac{t+g-1}{g} - k\right)_k \left(\frac{t+1}{g}\right)_{r-k}}{\left(\frac{2}{g}\right)_r}.$$

Theorem (B—, 2020).
Let W be a real coincidental reflection group of rank r. For $0 \leq k \leq r$, the following are equivalent as W-representations:

1. $\mathcal{V}^k(\mathcal{A})$, the k-th graded piece of the associated graded Varchenko-Gelfand ring
2. The σ_{r-k}-th eigenspace of the shuffle element $\mathcal{S}(W) \in \mathbb{R}W$
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To do: define the terms in blue!

Outline

Coincidental reflection groups

Notation:

$\mathcal{A}$ is an irreducible reflection (Coxeter) arrangement, i.e. an irreducible, real reflection group W acts on $\mathcal{A}$:

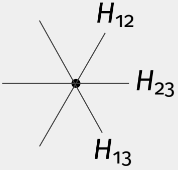
$$\text{reflection } s \in W \longleftrightarrow \text{hyperplane } H_s \in \mathcal{A}\,.$$

Assume $\mathcal{A}$ is central and has rank r.

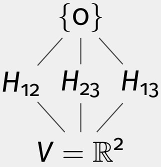
$\mathcal{L}(\mathcal{A})$ is the lattice of flats (intersection subspaces) of $\mathcal{A}$, ordered by reverse inclusion.

Example:

$\mathcal{A} =$



$\mathcal{L}(\mathcal{A}) =$



$W = S_3$

Coincidental reflection groups

W has exponents $e_1, e_2, \ldots, e_r$

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Definition

A reflection group is **coincidental** if its exponents form an arithmetic progression:

$$1, 1+g, 1+2g, \cdots, 1+(r-1)g.$$

for some integer g.

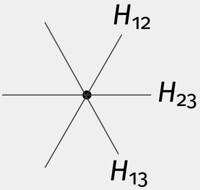
The *real* coincidental reflection groups are:

W	r:= rank	exponents	g:= progression
S_n	$n-1$	$1, 2, 3, \cdots, n-1$	1
B_n	n	$1, 3, 5, \cdots, 2n-1$	2
H_3	3	$1, 5, 9$	4
$I_2(m)$	2	$1, m-1$	$m-2$

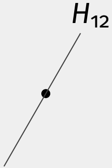
For $X \in \mathcal{L}(\mathcal{A})$, the restriction arrangement $\mathcal{A}^X$ is

$$\mathcal{A}^X := \{H \cap X : H \in \mathcal{A}, X \not\subset H\}.$$

$\mathcal{A} =$



$\mathcal{A}^{H_{12}} =$



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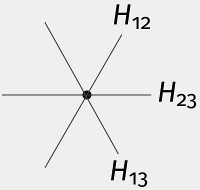
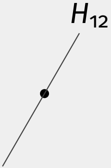
$\mathcal{A} =$ 

$\mathcal{A}^{H_{12}} =$ 

Theorem: (Abramenko, 1994; Aguiar-Mahajan, 2017).

$\mathcal{A}^X$ is a reflection arrangement for every $X \in \mathcal{L}(\mathcal{A})$

if and only if

W is a (product of) coincidental reflection group(s)

When W is coincidental: $\mathcal{A}^X \cong \mathcal{A}^Y$ if and only if $\dim(X) = \dim(Y)$

Outline

The Varchenko-Gelfand ring

Goal: Define an analogue of $H^* \operatorname{Conf}_n(\mathbb{R}^d)$ for any $\mathcal{A}$:

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Recall:

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Definition:

For any central hyperplane arrangement $\mathcal{A}$ of rank r,

$$\mathcal{M}^d(\mathcal{A}) := \mathbb{R}^{rd} \setminus \Big(\bigcup_{H_i\in\mathcal{A}} H_i \otimes \mathbb{R}^d \Big)$$

As in Type A, we are interested in $H^*(\mathcal{M}^d(\mathcal{A}))$.

The cohomology of $\mathcal{M}^d(\mathcal{A})$

When $d = 2$ (or any even number), as a graded ring:

$$H^*(\mathcal{M}^2(\mathcal{A})) \cong_W \text{ the Orlik-Solomon algebra of } \mathcal{A}\,.$$

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Theorem (Moseley, 2017).

When $d \geq 3$ and **odd**, there is a graded ring isomorphism

$$H^*(\mathcal{M}^d(\mathcal{A})) \cong_W \mathcal{V}(\mathcal{A}),$$
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where

$\mathcal{V}(\mathcal{A}) :=$the **associated graded Varchenko-Gelfand ring**

$\mathcal{V}^k(\mathcal{A}) :=$ the k-th graded piece of $\mathcal{V}(\mathcal{A})$.

The cohomology of $\mathcal{M}^d(\mathcal{A})$

When $d = 2$ (or any even number), as a graded ring:

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Intuition:

$\mathcal{V}(\mathcal{A})$ is a commutative version of the Orlik-Solomon algebra.

The Varchenko-Gelfand Ring

Definition/Theorem (Varchenko-Gelfand, 1987).

The **associated graded Varchenko-Gelfand ring** $\mathcal{V}(\mathcal{A})$ has presentation

$$\mathbb{R}[e_{H_i} : H_i \in \mathcal{A}]/\mathcal{J}$$

where $\mathcal{J}$ is generated by:

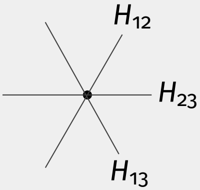
1. $e_{H_i}^2$ for each $H_i \in \mathcal{A}$;
2. For every circuit $C = C^+ \sqcup C^-$ in $\mathcal{A}$ with $C = (H_1, H_2, \cdots, H_m)$,

$$\sum_{i=1}^{m} c(i) e_{H_1} \cdots \widehat{e_{H_i}} \cdots e_{H_m}$$

where

$$c(i) = \begin{cases} 1 & \text{if } H_i \in C^-, \\ -1 & \text{if } H_i \in C^+. \end{cases}$$

Example: In Type A, when $n = 3$:

$\mathcal{A} =$ 

There is 1 circuit: $C^+ = H_{12}, H_{23}$ and $C^- = H_{13}$. Hence

$$\mathcal{V}(\mathcal{A}) = \mathbb{R}[e_{H_{12}}, e_{H_{23}}, e_{H_{13}}] / \Big\langle e_{H_{12}}^2, e_{H_{23}}^2, e_{H_{13}}^2, \\ e_{H_{12}}e_{H_{23}} - e_{H_{12}}e_{H_{13}} - e_{H_{23}}e_{H_{13}} \Big\rangle$$

Note: This matches Cohen's presentation of $H^* \operatorname{Conf}_3(\mathbb{R}^d)$, d odd

More generally,

In Type A, $\mathcal{V}(\mathcal{A})$ matches Cohen's presentation:

$$\mathcal{V}(\mathcal{A}) \longleftrightarrow H^* \operatorname{Conf}_n(\mathbb{R}^d), d \text{ odd}.$$

In Type B, $\mathcal{V}(\mathcal{A})$ matches Xicotencatl's (1997) presentation:

$$\mathcal{V}(\mathcal{A}) \longleftrightarrow H^* \operatorname{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^d), d \text{ odd},$$

where

$$\operatorname{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^d) := \{(x_1, \cdots, x_n) \in \mathbb{R}^{dn} : x_i \neq \pm x_j, x_i \neq 0\}$$

is the $\mathbb{Z}_2$-orbit configuration space of $\mathbb{R}^d$.

Outline

Eulerian idempotents

Recall that $\mathfrak{e}_k \in \mathbb{R}S_n$ were defined in two ways:

1. As the idempotent projectors onto the eigenspaces of the shuffle element $\mathcal{S}$, and
2. Via the generating function

$$\sum_{k=0}^{n-1} t^{k+1}\mathfrak{e}_k = \sum_{w\in S_n} \binom{t-1+n-\operatorname{des}(w)}{n} w.$$

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The Eulerian idempotents have been extensively studied and generalized since then!

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$$\sum_{k=0}^{n} t^k \mathfrak{e}_k = \sum_{w \in B_n} \binom{\frac{t-1}{2} + n - \mathrm{des}(w)}{n} w.$$

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1992: Bergeron-Bergeron-Howlett-Taylor define a finer family of idempotents in $\mathcal{D}(W)$ for any reflection group W

The idempotents are indexed by descent sets; summing over idempotents with the same descent size recovers the $\mathfrak{e}_k$

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2017: Aguiar-Mahajan further develop the theory of $\mathfrak{e}_X$, particularly for coincidental reflection groups

Upshot:

For any reflection group, these definitions all recover the same family of idempotents $\mathfrak{e}_k \in \mathbb{R}W$ for $0 \leq k \leq r$...

Call this family the **Eulerian idempotents**.

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There is yet *another* way to define these Eulerian idempotents...

A generalized shuffle element

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Barr's Shuffle element can be rephrased in terms of descents:

$$\mathcal{S} := \sum_{i=1}^{n-1} \sum_{\substack{w \in S_n: \\ w_1 < \cdots < w_i \\ w_{i+1} < \cdots < w_n}} w = \sum_{i=1}^{n-1} \sum_{\substack{w \in S_n: \\ \mathrm{Des}(w) \subset \{i\}}} w \in \mathcal{D}(S_n) \subset \mathbb{R} S_n.$$

Example: When $n = 3$,

$$\mathcal{S} = \underbrace{(\mathbf{1},\mathbf{2},\mathbf{3}) + (\mathbf{2},\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{1},\mathbf{2})}_{\mathrm{Des}(w)\subset\{1\}} + \underbrace{(\mathbf{1},\mathbf{2},\mathbf{3}) + (\mathbf{1},\mathbf{3},\mathbf{2}) + (\mathbf{2},\mathbf{3},\mathbf{1})}_{\mathrm{Des}(w)\subset\{2\}}$$

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Definition (B—, 2020). For any reflection group W with generators $s_1, \cdots, s_r$, the **shuffle element** $\mathcal{S}(W)$ is defined by

$$\mathcal{S}(W) := \sum_{i=1}^{r} \sum_{\substack{w \in W: \\ \mathrm{Des}(w) \subset \{s_i\}}} w \in \mathcal{D}(W) \subset \mathbb{R}\, W.$$

Proposition (B—, 2020).

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- When W is coincidental, $\mathcal{S}(W)$ has $r + 1$ distinct integer eigenvalues $\sigma_0 < \sigma_1 < \cdots < \sigma_r$ and,
- When W is coincidental, the projector onto the σ_k-th eigenspace of $\mathcal{S}(W)$ recovers the Eulerian idempotents $\mathfrak{e}_k$.

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When W is coincidental, $\mathcal{S}(W)$ has $r+1$ distinct integer eigenvalues $\sigma_0 < \sigma_1 < \cdots < \sigma_r$ and,

When W is coincidental, the projector onto the σ_k-th eigenspace of $\mathcal{S}(W)$ recovers the Eulerian idempotents $\mathfrak{e}_k$.

This allows us to generalize the Eulerian subalgebra:

Theorem (B—, 2020).

There is an **Eulerian subalgebra** of $\mathcal{D}(W)$ generated by sums of elements with the same descent number *if and only if* W is coincidental. This subalgebra is always commutative.

Main Results

Recall the rising factorial $(t)_k := (t)(t+1)\dots(t+k-1)$ and let

$$\beta_{W,k}(t) := \frac{\left(\frac{t+g-1}{g} - k\right)_k \left(\frac{t+1}{g}\right)_{r-k}}{\left(\frac{2}{g}\right)_r}.$$

Theorem (B—, 2020).
Let W be a real coincidental reflection group of rank r. For $0 \leq k \leq r$, the following are equivalent as W-representations:

1. $\mathcal{V}^k(\mathcal{A})$, the k-th graded piece of the associated graded Varchenko-Gelfand ring
2. The σ_{r-k}-th eigenspace of the shuffle element $\mathcal{S}(W) \in \mathbb{R}W$
3. The representation $\mathbb{R}W\mathfrak{e}_{r-k}$ where $\mathfrak{e}_{r-k}$ is defined by

$$\sum_{k=0}^{r} t^k \mathfrak{e}_k = \sum_{w \in W} \beta_{W,\mathrm{des}(w)}(t) \cdot w.$$

Big idea:

Map $\mathcal{S}(W)$ into the Tits (face) semigroup algebra of $\mathcal{A}$

Relate eigenvalues of $\mathcal{S}(W)$ to restriction arrangements $\mathcal{A}^X$

Use the fact that when W is coincidental, $\mathcal{A}^X$ depends only on the dimension of X

Upshot: This allows for a uniform, character-free argument!

Thank you for **listening**!

Future Directions

Complex Reflection Groups:

There are complex (non-real) coincidental reflection groups

These are precisely Shephard groups, which are the symmetry groups of complex polytopes

Question: To what extent does the story of the real Eulerian representations generalize to Shephard groups?

I would love to discuss any ideas in this direction!

Properties of the Eulerian representations

Many representation theoretic properties of $\mathfrak{e}_k$ in Type A are not known in other types!

Currently: $\mathbb{R} S_n \mathfrak{e}_k$ has a “hidden” S_{n+1} action. I am working on generalizing this to type B using configuration spaces