Enumerating Linear Systems on Graphs

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(Joint with David Perkinson and Forrest Glebe)
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AMS Special Session on Combinatorial Algebraic Geometry

Outline

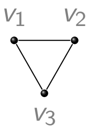
- **Background:** Divisors on graphs
- **Problem:** Classify complete linear systems of divisors
- **Solutions:**
  1. Primary and secondary divisors
  2. Integer points in polyhedra
  3. Invariant theory

Background: Divisors on graphs

Example:

- $G = (V, E)$ is a connected graph

C_3 :



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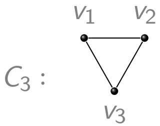

- The group of all divisors on G is $\mathrm{Div}(G) := \mathbb{Z}V \cong \mathbb{Z}^n$

$\mathrm{Div}(C_3) = \mathbb{Z}\{v_1, v_2, v_3\}$

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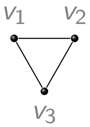

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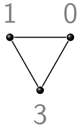
- A *divisor* on G is an element $D \in \mathrm{Div}(G)$



Background: Divisors on graphs

- Write $D \in \text{Div}(G)$ and $D(v_i) \in \mathbb{Z}$ as $D = D(v_1)v_1 + \cdots + D(v_n)v_n$

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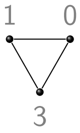

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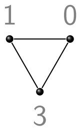

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- The *degree* of D is $\deg(D) = \sum_{v \in V} D(v)$

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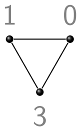

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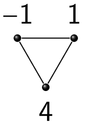
- *Linear equivalence* of divisors is determined by the *Laplacian*

$$L : \mathbb{Z}^n \to \mathbb{Z}^n$$

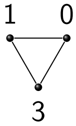
$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Example: linear equivalence



$-v_1 + v_2 + 4v_3$

$\sim$



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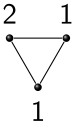
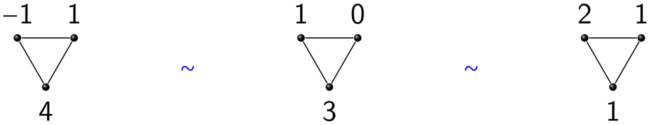


$2v_1 + v_2 + v_3$

Background: Divisors on graphs

A divisor D is *effective* if $D(v) \geq 0$ for all $v \in V$



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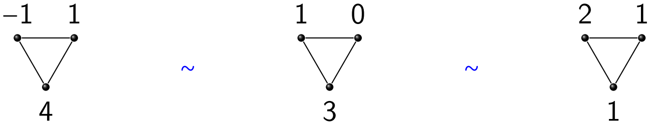

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Complete linear system for $D \in \mathrm{Div}(G)$:

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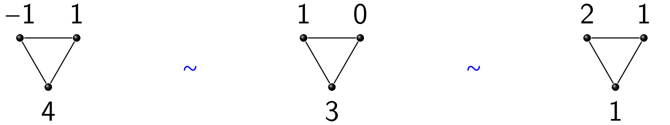

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Goal: Characterize $|D|$ for any graph G and divisor $D \in \mathrm{Div}(G)$.

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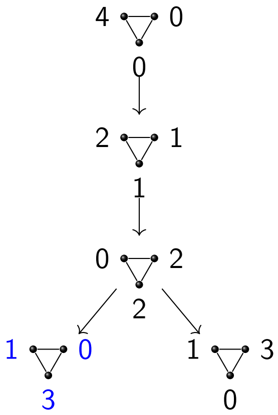

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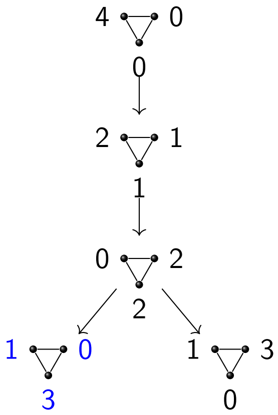

The complete linear system of $v_1 + 3v_3$ is

$$|v_1 + 3v_3| = \{4v_1, 2v_1 + v_2 + v_3, 2v_2 + 2v_3, v_1 + 3v_2, v_1 + 3v_3\}$$

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Approach: Partition the set of all effective divisors

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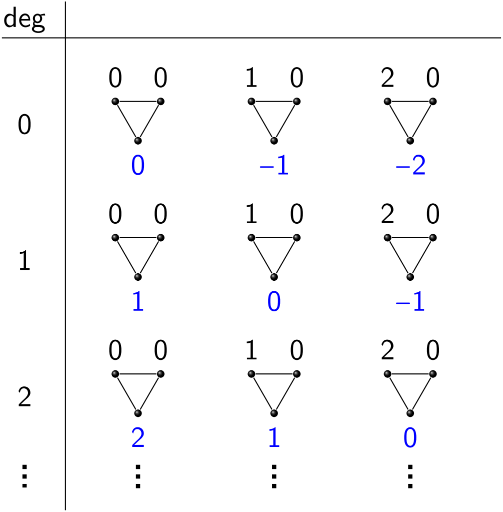
$$\text{e.g. } \mathrm{Jac}(C_3) \cong \mathbb{Z}/3\mathbb{Z}$$

Problem: Classify complete linear systems

Fix $q = v_3$.

Enumerate $\mathrm{Pic}^+(C_3)$ by degree using $\mathrm{Jac}(C_3) \cong \mathbb{Z}/3\mathbb{Z}$:



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Pic$^+(G)$

deg					
0	$[0]$	$[D_2]$	$\dots$	$[D_\kappa]$	$\leftarrow$ Jac(G)
1	$[q]$	$[D_2 + q]$	$\dots$	$[D_\kappa + q]$	
2	$[2q]$	$[D_2 + 2q]$	$\dots$	$[D_\kappa + 2q]$	
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

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$$
\mathrm{Pic}^{+}(G) \quad
\begin{array}{c|cccc}
\deg & & & & \\
\hline
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\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\quad \leftarrow \ \mathrm{Jac}(G)
$$

Underlying idea:

$$
\begin{aligned}
\mathrm{Pic}(G) &\xrightarrow{\sim} \mathrm{Jac}(G) \oplus \mathbb{Z} \\
[D] &\mapsto ([D - \deg(D) q], \deg(D)).
\end{aligned}
$$

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$\mathrm{Pic}^+(G)$ (rows)

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Related Goal: Compute $\Lambda_{[D]}(z) := \sum_{k=0}^{\infty} \#|D + kq| z^k$

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For every graph G there is a finite set

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such that each $E \in \mathbb{E}_{[D]}$ can be written uniquely as

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with $F \in \mathcal{S}_{[D]}$ and $a_P \in \mathbb{Z}_{\geq 0}$ for all $P \in \mathcal{P}$.

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Corollary.

$$\Lambda_{[D]}(z) := \sum_{k=0}^{\infty} \#|D + kq| z^k = \frac{\sum_{F \in \mathcal{S}_{[D]}} z^{\deg(F)}}{\prod_{P \in \mathcal{P}} (1 - z^{\deg(P)})}$$

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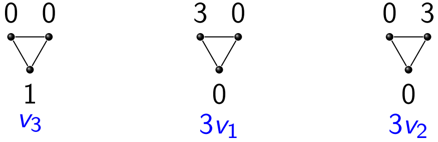
Primary divisors $\mathcal{P}$ for C_3 with $q = v_3$:



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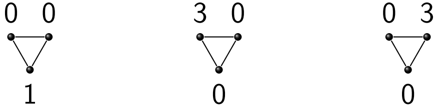

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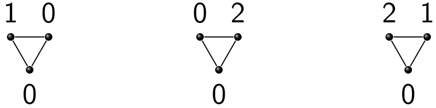

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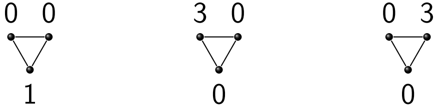

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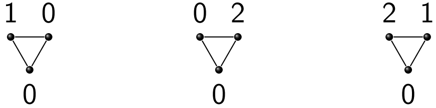

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Note that $v_1 + 3v_3 \in \mathbb{E}_{[D]}$ and

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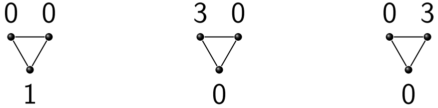

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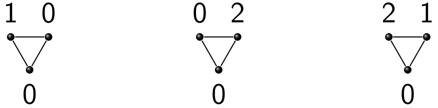

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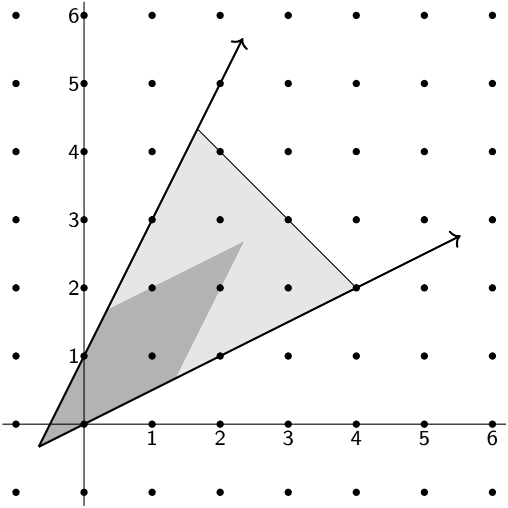
Solution 2: Integer points of polyhedra

Polyhedra



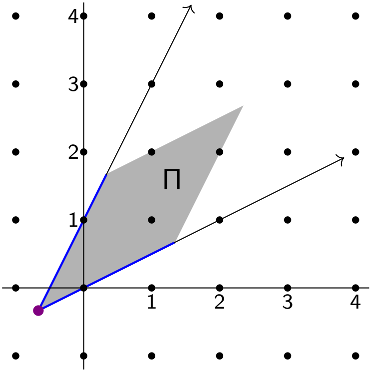
Solution 2: Integer points of polyhedra

A *rational simplicial pointed cone*

$$\mathcal{K} = \{p + \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n : \lambda_1, \ldots, \lambda_n \geq 0\}$$

generating rays = $\{\omega_1, \ldots, \omega_n\} \subset \mathbb{Z}^n$

fundamental parallelepiped $= \{\lambda_1, \ldots, \lambda_n : 1 > \lambda_1, \ldots, \lambda_n \geq 0\}$



Solution 2: Integer points of polyhedra

Theorem. (B, Glebe, Perkinson) For every $[D] \in \mathrm{Jac}(G)$,

there is a rational simplicial pointed cone $\mathcal{K}_D$ and **bijections**

$$\mathbb{E}_{[D]} \longleftrightarrow \text{lattice points of } \mathcal{K}_D$$

$$\text{primary divisors } \mathcal{P} \longleftrightarrow \text{generating rays } \{\omega_1, \ldots, \omega_n\}$$

$$\text{secondary divisors } \mathcal{S}_{[D]} \longleftrightarrow \text{lattice points of fundamental parallelepiped}$$

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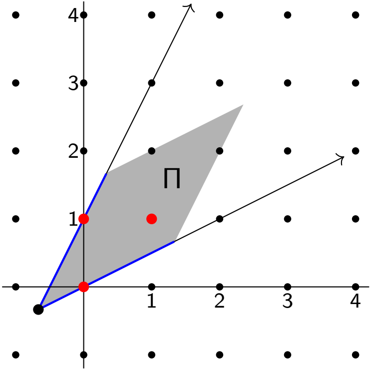
Corollary.

Integer-point transform of $\mathcal{K}_D$ rediscovers

$$\Lambda_{[D]}(z) := \sum_{k=0}^{\infty} \#|D + kq| z^k = \frac{\sum_{F \in \mathcal{S}_{[D]}} z^{\deg(F)}}{\prod_{P \in \mathcal{P}} (1 - z^{\deg(P)})}$$

Example: Integer points of polyhedra

Let $[D] = [v_1 - v_3] \in \mathrm{Jac}(C_3)$.
Projecting $\mathcal{K}_D \subset \mathbb{R}^3$ to a polyhedra in $\mathbb{R}^2$ gives



Example: Integer points of polyhedra

Recall that

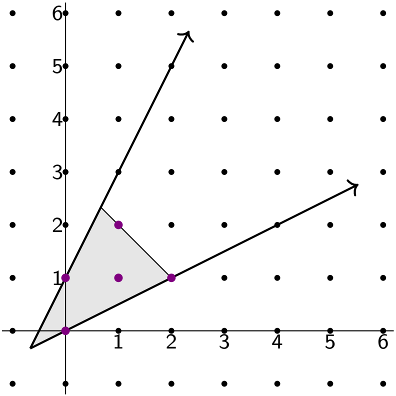
$$\left|v_1 + 3v_3\right| = \left\{4v_1, 2v_1 + v_2 + v_3, 2v_2 + 2v_3, v_1 + 3v_2, v_1 + 3v_3\right\}$$

Example: Integer points of polyhedra

Recall that

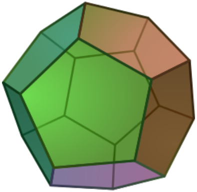
$$|v_1 + 3v_3| = \{4v_1, 2v_1 + v_2 + v_3, 2v_2 + 2v_3, v_1 + 3v_2, v_1 + 3v_3\}$$

$|v_1 + 3v_3|$ can be identified with integer points in the polytope:



Solution 3: Invariant theory

$$\Phi_{\Gamma,\chi}(z) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\overline{\chi(\gamma)}}{\det(I_n - z\gamma)}.$$



Invariant Theory

$$a_{(\lambda_1+n-1,\lambda_2+n-2,\dots,\lambda_n)}(x_1,x_2,\dots,x_n) = \det\begin{bmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \dots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \dots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \dots & x_n^{\lambda_n} \end{bmatrix}$$

Solution 3: Invariant theory

Theorem (B, Glebe, Perkinson)

For any G, there is a representation $\rho : \mathrm{Jac}(G)^* \longrightarrow \mathrm{GL}(\mathbb{C}^n)$ with

Action: $\Gamma := \rho(\mathrm{Jac}(G)^*)$ acts on $\mathbb{C}[\mathbf{x}]$

Character: For every $[D] \in \mathrm{Jac}(G)$

$$\begin{aligned}[D] : \Gamma &\longrightarrow \mathbb{C}^\times \\ \rho(\varphi) &\mapsto \varphi([D])\end{aligned}$$

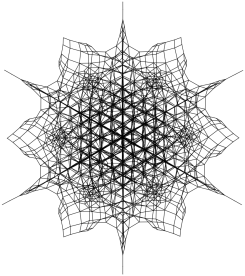
Such that for every $[D] \in \mathrm{Jac}(G)$, there are **bijections**

$$\mathbb{E}_{[D]} \longleftrightarrow \text{monomial } \mathbb{C}\text{-basis for } \mathbb{C}[\mathbf{x}]^\Gamma_{[D]}$$

$$\text{primary divisors } \mathcal{P} \longleftrightarrow \text{monomial primary invariants in } \mathbb{C}[\mathbf{x}]^\Gamma$$

$$\text{secondary divisors } \mathcal{S}_{[D]} \longleftrightarrow \text{monomial [D]-relative invariants in } \mathbb{C}[\mathbf{x}]^\Gamma_{[D]}$$

Thanks!



References

"Enumerating Linear Systems on Graphs," (2019).
S. Brauner, F. Glebe, D. Perkinson, arXiv:1906.04768.

Example: Invariant theory

For $G = C_3$, $\text{Jac}(C_3)^* \cong \mathbb{Z}/3\mathbb{Z} = \langle \varphi \rangle$.

$\rho : \text{Jac}(C_3)^* \to GL_3(\mathbb{C})$ is the regular representation of $\mathbb{Z}/3\mathbb{Z}$:

$$\rho : \varphi \mapsto \begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Minimal monomial primary invariants are x_1^3, x_2^3 and x_3.

Compare to the primary divisors $\mathcal{P}$ for C_3: