Eulerian representations for Coincidental reflection groups

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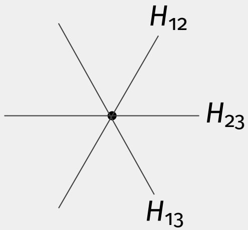


Outline

Big Idea:

Generalize a beautiful Type *A* story connecting combinatorics, representation theory and topology to a broader class of reflection groups

Outline:

1. Motivating Story: Type A
2. Coincidental reflection groups
3. Eulerian idempotents
4. The Varchenko-Gelfand ring
5. Main Results

Motivating Story: Type A

Descents

The story of the Eulerian idempotents begins with descents...

For $w = (w_1, \cdots, w_n) \in S_n$, the **descent set** of w is

$$\mathrm{Des}(w) := \{i \in [n-1] : w_i > w_{i+1}\}$$

The **descent number** of w is $\mathrm{des}(w) := \# \mathrm{Des}(w)$.

Example: If $w = (1, \mathbf{4}, 2, \mathbf{5}, 3)$, then $\mathrm{Des}(w) = \{\mathbf{2}, \mathbf{4}\}$ and $\mathrm{des}(w) = 2$.

Equivalently, in the language of Coxeter groups:

$$\mathrm{Des}(w) := \Big\{ \underbrace{s_i}_{\substack{\text{transposition} \\ (i,i+1)}} \in \underbrace{S}_{\substack{\text{Coxeter} \\ \text{generators}}} : \underbrace{\ell(ws_i) < \ell(w)}_{\text{Coxeter length}} \Big\}.$$

Remark: **Des** and **des** can be defined for any Coxeter group.

Let $\mathbb{R}\,S_n$ be the group algebra of S_n and $w = (w_1, w_2, \cdots w_n) \in S_n$.

Surprising fact: (Solomon, 1976)

There is a *subalgebra* of $\mathbb{R}\,S_n$ generated by sums of elements with the same descent set:

$$\mathcal{D}(S_n) := \left\langle Y_T := \sum_{\substack{w \in S_n \\ \mathrm{Des}(w)=T}} c_T w \,:\, c_T \in \mathbb{R}, T \subset [n-1] \right\rangle$$

called **Solomon's descent algebra.**

Example: When $n = 3$, the descent algebra $\mathcal{D}(S_3)$ has basis:

$$\begin{aligned} Y_\emptyset &= (1,2,3) \\ Y_1 &= (2,1,3) + (3,1,2) \\ Y_2 &= (1,3,2) + (2,3,1) \\ Y_{1,2} &= (3,2,1). \end{aligned}$$

 Remark: In fact, $\mathcal{D}(W)$ is a subalgebra for any Coxeter group.

Theorem (Garsia-Reutenauer, 1989).

There is a family of idempotents in $\mathbb{R}\, S_n$ defined by

$$\sum_{k=0}^{n-1} t^{k+1} \mathfrak{e}_k = \sum_{w \in S_n} \binom{t-1+n-\operatorname{des}(w)}{n} w.$$

Call this family the **Eulerian idempotents**.

Remark:

By construction, the $\mathfrak{e}_k$ are in the Descent algebra $\mathcal{D}(S_n)$

In fact, the $\mathfrak{e}_k$ generate a commutative subalgebra of $\mathcal{D}(S_n)$ spanned by sums of elements with the same descent number

This subalgebra is known as the **Eulerian subalgebra**.

Example: When $n = 3$,

$$\mathfrak{e}_0 = \frac{1}{6}\big((1,2,3) - (2,1,3) - (3,1,2) - (1,3,2) - (2,3,1) + 2(3,2,1)\big)$$
$$= \frac{1}{6}\big(Y_\emptyset - Y_1 - Y_2 + 2Y_{1,2}\big)$$

$$\mathfrak{e}_1 = \frac{1}{2}\big((1,2,3) - (3,2,1)\big)$$
$$= \frac{1}{2}\big(Y_\emptyset - Y_{1,2}\big)$$

$$\mathfrak{e}_2 = \frac{1}{6}\big((1,2,3) + (2,1,3) + (3,1,2) + (1,3,2) + (2,3,1) + (3,2,1)\big)$$
$$= \frac{1}{6}\big(Y_\emptyset + Y_1 + Y_2 + Y_{1,2}\big)$$

Definition (Barr, 1968).

The **Shuffle (Barr) element** in $\mathbb{R} S_n$ is

$$\mathcal{S} := \sum_{i=1}^{n-1} \sum_{\substack{w \in S_n: \\ \mathrm{Des}(w) \subset \{i\}}} w \in \mathcal{D}(S_n) \subset \mathbb{R} S_n.$$

Example: When $n = 3$,

$$\mathcal{S} = \underbrace{(\mathbf{1},\mathbf{2},\mathbf{3}) + (\mathbf{2},\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{1},\mathbf{2})}_{\mathrm{Des}(w)\subset\{1\}} + \underbrace{(\mathbf{1},\mathbf{2},\mathbf{3}) + (\mathbf{1},\mathbf{3},\mathbf{2}) + (\mathbf{2},\mathbf{3},\mathbf{1})}_{\mathrm{Des}(w)\subset\{2\}}$$
$$= 2(1,2,3) + (2,1,3) + (3,1,2) + (1,3,2) + (2,3,1).$$

Theorem (Gerstenhaber-Schack, 1987).

$\mathcal{S}$ acts semisimply on $\mathbb{R}\, S_n$

$\mathcal{S}$ has eigenvalues $\sigma_k := 2^{k+1} - 2$ for $0 \leq k \leq n-1$.

Corollary.

By Lagrange interpolation, the idempotent projecting onto the σ_k-th eigenspace of $\mathcal{S}$ is

$$\mathfrak{e}_k := \prod_{j \neq k} \frac{\mathcal{S} - \sigma_j}{\sigma_k - \sigma_j}.$$

Theorem (Loday, 1989).

These idempotents are precisely the Eulerian idempotents

Eulerian idempotents for $n = 3$

Example: When $n = 3$, the Barr element $\mathcal{S}$ has eigenvalues $0, 2, 6$:

$$\mathfrak{e}_0 = \frac{(\mathcal{S}-2)(\mathcal{S}-6)}{(0-2)(0-6)} = \frac{1}{6}\big((1,2,3) - (2,1,3) - (3,1,2) - (1,3,2) - (2,3,1) + 2(3,2,1)\big)$$

$\sigma_0 = 0$-eigenspace projector

$$\mathfrak{e}_1 = \frac{(\mathcal{S}-0)(\mathcal{S}-6)}{(2-0)(2-6)} = \frac{1}{2}\big((1,2,3) - (3,2,1)\big)$$

$\sigma_1 = 2$-eigenspace projector

$$\mathfrak{e}_2 = \frac{(\mathcal{S}-0)(\mathcal{S}-2)}{(6-0)(6-2)} = \frac{1}{6}\big((1,2,3) + (2,1,3) + (3,1,2) + (1,3,2) + (2,3,1) + (3,2,1)\big)$$

$\sigma_2 = 6$-eigenspace projector

S_n acts on acts on $\mathbb{R}\, S_n$ and $\mathbb{R}\, S_n \mathfrak{e}_k$ by left multiplication...

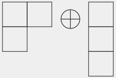
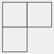
The **Eulerian representations** are defined as the family of representations $\mathbb{R}\, S_n \mathfrak{e}_k$ induced by this action

By construction, $\mathbb{R}\, S_n \mathfrak{e}_k$ is the σ_k-eigenspace of $\mathcal{S}$

Example: When $n = 3$ for any $\tau \in S_3$,

$$\tau \cdot \mathfrak{e}_2 = \tau \cdot \frac{1}{6} \sum_{\sigma \in S_3} \sigma = \frac{1}{6} \sum_{\sigma \in S_3} \tau\sigma = \frac{1}{6} \sum_{\sigma' \in S_3} \sigma' = \mathfrak{e}_2.$$

Example: $n = 3$

k	Eulerian representation	Irreducible Decomposition
2	$\mathbb{R}S_3\mathfrak{e}_2 = \sigma_2$-eigenspace	
1	$\mathbb{R}S_3\mathfrak{e}_1 = \sigma_1$-eigenspace	
0	$\mathbb{R}S_3\mathfrak{e}_0 = \sigma_0$-eigenspace	

Question:

Where do these representations naturally appear?

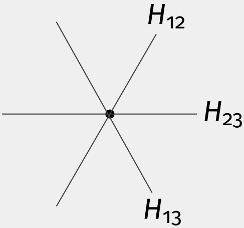
The answer is closely related to the **braid arrangement**,

$$\mathcal{A}_{S_n} := \{H_{ij} : 1 \le i < j \le n\}$$

where

$$H_{ij} := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i = x_j\}.$$

Example. When $n = 3$, the essentialized braid arrangement $\mathcal{A}_{S_3}$ is



The braid arrangement has **complement**

$$\mathcal{M}(\mathcal{A}_{S_n}) := \mathbb{R}^n \setminus \mathcal{A} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \neq x_j \text{ for } i, j \in [n]\}$$
$$= \text{the } n\text{-th ordered configuration space of } \mathbb{R}$$
$$= \mathrm{Conf}_n(\mathbb{R}).$$

We are interested in the ***d*-thickened complement**

$$\mathcal{M}^d(\mathcal{A}_{S_n}) := \mathcal{M}(\mathcal{A}) \otimes \mathbb{R}^d = \mathbb{R}^{dn} \setminus \Big(\bigcup_{1 \le i < j \le n} H_{ij} \otimes \mathbb{R}^d \Big)$$
$$= \{(x_1, \cdots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j \text{ for } i, j \in [n]\}$$
$$= \text{the } n\text{-th ordered configuration space of } \mathbb{R}^d$$
$$= \mathrm{Conf}_n(\mathbb{R}^d).$$

Example: When $d = 2$, this is equivalent to the complement of the complexified arrangement $\mathcal{M}(\mathcal{A}) \otimes \mathbb{C}$

A natural question:

$$\text{what is } H^*(\mathrm{Conf}_n(\mathbb{R}^d), \mathbb{R})?$$

Theorem (Arnol'd (1969): $d = 2$, F. Cohen (1976): $d \geq 2$).

The ring $H^* \, \mathrm{Conf}_n(\mathbb{R}^d)$ has presentation

$$\mathbb{R}\langle e_{ij} : 1 \leq i \neq j \leq n \rangle / \mathcal{J}$$

where each e_{ij} is in degree $d-1$ and $\mathcal{J}$ is generated by

1. e_{ij}^2
2. $e_{ij} = (-1)^d e_{ji}$
3. $e_{ij}e_{j\ell} + e_{j\ell}e_{\ell i} + e_{\ell i}e_{ij}$

for any $1 \leq i \neq j \neq \ell \leq n$.

This implies that $H^* \, \mathrm{Conf}_n(\mathbb{R}^d)$ is

concentrated in degrees $k(d-1)$ for $0 \leq k \leq n-1$

commutative when d is **odd**

anti-commutative when d is **even**

The symmetric group S_n acts on

$$\mathrm{Conf}_n(\mathbb{R}^d) = \{(x_1 \cdots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j \text{ for } i, j \in [n]\},$$

making $H^* \mathrm{Conf}_n(\mathbb{R}^d)$ into an S_n-module...

Known fact: When d is **odd**,

$$H^* \mathrm{Conf}_n(\mathbb{R}^d) \cong_{S_n} \mathbb{R} S_n.$$

A more refined question:

What representation does $H^{k(d-1)} \mathrm{Conf}_n(\mathbb{R}^d)$ carry for **each** k?

Example: $H^0 \mathrm{Conf}_n(\mathbb{R}^d)$ is always the trivial representation.

Key connection:

When $d \geq 3$ is **odd**, for $0 \leq k \leq n-1$,

$$H^{k(d-1)} \operatorname{Conf}_n(\mathbb{R}^d) \cong_{S_n} \mathbb{R}\, S_n \mathfrak{e}_{n-1-k}.$$

How do we know?

1990 Hanlon computes the characters of $\mathbb{R}\, S_n \mathfrak{e}_{n-1-k}$

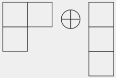
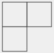
1997 Sundaram-Welker prove an equivariant formulation of the Goresky-MacPherson formula relating

cohomology of a subspace arrangement $\longleftrightarrow$ homology of its intersection lattice

As a special case:
they compute the characters of $H^{k(d-1)} \operatorname{Conf}_n(\mathbb{R}^d)$

Example: $n = 3$

Example: When $n = 3$ and d is odd,

Eulerian representation	Configuration space cohomology	Irreducible decomposition
$\mathbb{R}S_3\mathfrak{e}_2 = \sigma_2$-eigenspace	$H^0\,\mathrm{Conf}_3(\mathbb{R}^d)$ $= \mathbb{R}\{1\}$	
$\mathbb{R}S_3\mathfrak{e}_1 = \sigma_1$-eigenspace	$H^{1(d-1)}\,\mathrm{Conf}_3(\mathbb{R}^d)$ $= \mathbb{R}\{e_{12}, e_{23}, e_{13}\}/\mathcal{J}_1$	
$\mathbb{R}S_3\mathfrak{e}_0 = \sigma_0$-eigenspace	$H^{2(d-1)}\,\mathrm{Conf}_3(\mathbb{R}^d)$ $= \mathbb{R}\{e_{12}e_{23}, e_{12}e_{13}\}/\mathcal{J}_2$	

Summary: For $0 \leq k \leq n-1$, the following are equivalent as S_n-representations:

1. $H^{k(d-1)} \operatorname{Conf}_n(\mathbb{R}^d)$ for odd $d \geq 3$;
2. The k-th graded piece of Cohen's algebra for odd $d \geq 3$:
$$\mathbb{R}\langle e_{ij} : 1 \leq i < j \leq n \rangle / \mathcal{J}$$
3. The $\sigma_{n-1-k} = \{2^{n-k} - 2\}$-eigenspace of the Barr's shuffle element $\mathcal{S} \in \mathbb{R} S_n$;
4. The representation $\mathbb{R} S_n \mathfrak{e}_{n-1-k}$, where $\mathfrak{e}_{n-1-k}$ is defined by
$$\sum_{k=0}^{n-1} t^{k+1} \mathfrak{e}_k = \sum_{w \in S_n} \binom{t - 1 + n - \operatorname{des}(w)}{n} w.$$

Goal:

Generalize this statement to coincidental reflection groups,
i.e. reflection groups whose exponents form an arithmetic progression

$1, 1+g, 1+2g, 1+3g, \cdots$

Recall the rising factorial $(t)_k := (t)(t+1)\dots(t+k-1)$ and let

$$\beta_{W,k}(t) := \frac{\left(\frac{t+g-1}{g} - k\right)_k \cdot \left(\frac{t+1}{g}\right)_{r-k}}{\left(\frac{2}{g}\right)_r}.$$

Theorem (B—, 2020).
Let W be a real coincidental reflection group of rank r. For $0 \leq k \leq r$, the following are equivalent as W-representations:

1. $H^{k(d-1)}\mathcal{M}^d(\mathcal{A}_W)$ for odd $d \geq 3$
2. $\mathcal{V}^k(\mathcal{A}_W)$, the k-th graded piece of the associated graded Varchenko-Gelfand ring
3. The σ_{r-k}-th eigenspace of the shuffle element $\mathcal{S}(W) \in \mathbb{R}W$
4. The representation $\mathbb{R}W\mathfrak{e}_{r-k}$ where $\mathfrak{e}_{r-k}$ is defined by

$$\sum_{k=0}^{r} t^k \mathfrak{e}_k = \sum_{w \in W} \beta_{W,\mathrm{des}(w)}(t) \cdot w.$$

Outline

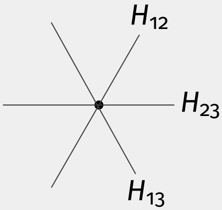
Coincidental reflection groups

Every Coxeter group has a **reflection arrangement** $\mathcal{A}_W$ where

$$\text{reflections } s \in W \longleftrightarrow \text{hyperplanes } H_s \in \mathcal{A}_W .$$

Example: The symmetric group S_3 acts on

$\mathcal{A}_{S_3} =$



The transposition $(ij) \in S_n$ reflects over the hyperplane H_{ij}

Exponents

Every Coxeter group W of rank r has a unique set of integers

$$e_1 = 1 \leq e_2 \leq \cdots \leq e_r$$

called the **exponents** of W, which satisfy many **product formulas**:

Statistic	$S_n = A_{n-1}$	W
exponents	$1, 2, \cdots, n-1$	$e_1, e_2, \cdots, e_r$
$\#W$	$n! = 2 \cdot 3 \cdots n$	$\prod_{i=1}^{r}(1+e_i)$
$\sum_{w \in W} q^{\ell(w)}$	$[n]_q! = [2]_q \cdot [3]_q \cdots [n]_q$	$\prod_{i=1}^{r} \frac{q^{1+e_i}-1}{q-1}$
$\sum_{w \in W} q^{\dim(V^w)}$	$(q+1)(q+2)\cdots(q+n-1)$	$\prod_{i=1}^{r}(q+e_i)$
$\sum_{X \in \mathcal{L}(\mathcal{A}_W)} \mu(V,X) q^{\dim(X)}$	$(q-1)(q-2)\cdots(q-n+1)$	$\prod_{i=1}^{r}(q-e_i)$

W has exponents $e_1, e_2, \ldots, e_r$.

Definition

A reflection group is **coincidental** if its exponents form an arithmetic progression:

$$1, 1+g, 1+2g, \cdots, 1+(r-1)g.$$

for some integer g.

The *real* coincidental reflection groups are:

W	r:= rank	exponents	g:= progression
S_n	$n-1$	$1, 2, 3, \cdots, n-1$	1
B_n	n	$1, 3, 5, \cdots, 2n-1$	2
H_3	3	$1, 5, 9$	4
$I_2(m)$	2	$1, m-1$	$m-2$

Eulerian idempotents

Recall that $\mathfrak{e}_k \in \mathbb{R}\, S_n$ were defined in two ways:

1. As the idempotent projectors onto the eigenspaces of the shuffle element $\mathcal{S}$, and
2. Via the generating function

$$\sum_{k=0}^{n-1} t^{k+1}\mathfrak{e}_k = \sum_{w \in S_n} \binom{t-1+n-\operatorname{des}(w)}{n} w.$$

The Eulerian idempotents have been extensively studied and generalized since then!

1992: Bergeron-Bergeron define a Type B analog:

$$\sum_{k=0}^{n} t^k \mathfrak{e}_k = \sum_{w \in B_n} \binom{\frac{t-1}{2} + n - \operatorname{des}(w)}{n} w.$$

1992: Bergeron-Bergeron-Howlett-Taylor define a finer family of idempotents in $\mathcal{D}(W)$ for any reflection group W

The idempotents are indexed by descent sets; summing over idempotents with the same descent size recovers the $\mathfrak{e}_k$

2009: Saliola constructs for any central arrangement $\mathcal{A}$, a family of idempotents $\mathfrak{e}_X$ for each flat $X \in \mathcal{L}(\mathcal{A})$

In the case that $\mathcal{A}$ *is a reflection arrangement, the* $\mathfrak{e}_X$ *can be realized in* $\mathbb{R}W$

2017: Aguiar-Mahajan further develop the theory of $\mathfrak{e}_X$, particularly for coincidental reflection groups

Generalizing the Eulerian idempotents

Upshot:

For any reflection group, these definitions all recover the same family of idempotents $\mathfrak{e}_k \in \mathbb{R}W$ for $0 \leq k \leq r$...

Call this family the **Eulerian idempotents**.

Recall how Barr's shuffle element was defined:

$$\mathcal{S} := \sum_{i=1}^{n-1} \sum_{\substack{w \in S_n: \\ \mathrm{Des}(w) \subset \{i\}}} w \in \mathcal{D}(S_n) \subset \mathbb{R} S_n.$$

Definition (B—, 2020). For any reflection group W with generators $s_1, \cdots, s_r$, the **shuffle element** $\mathcal{S}(W)$ is defined by

$$\mathcal{S}(W) := \sum_{i=1}^{r} \sum_{\substack{w \in W: \\ \mathrm{Des}(w) \subset \{s_i\}}} w \in \mathcal{D}(W) \subset \mathbb{R} W.$$

Example: In B_2 with Coxeter generators s and t,

$$\mathcal{S}(B_2) = \underbrace{1 + s + ts + sts}_{\mathrm{Des}(w) \subset \{s\}} + \underbrace{1 + t + st + tst}_{\mathrm{Des}(w) \subset \{t\}}$$

Proposition (B—, 2020).

$\mathcal{S}(W)$ acts semisimply on $\mathbb{R}W$ for any reflection group W.

When W is coincidental,

$\mathcal{S}(W)$ has $r+1$ distinct, non-negative, integer eigenvalues $\sigma_0 < \sigma_1 < \cdots < \sigma_r$ and,

the projector onto the σ_k-th eigenspace of $\mathcal{S}(W)$ recovers the Eulerian idempotents.

This allows us to generalize the Eulerian subalgebra:

Theorem (B—, 2020).

There is an **Eulerian subalgebra** of $\mathcal{D}(W)$ generated by sums of elements with the same descent number *if and only if* W is coincidental. This subalgebra is always commutative.

Outline

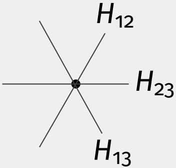
The Varchenko-Gelfand ring

Varchenko and Gelfand define **Heaviside functions** on $\mathcal{M}(\mathcal{A})$ by

$$e_{H_i}(v) = \begin{cases} 1 & v \in H_i^+ \\ 0 & v \in H_i^- \end{cases}$$

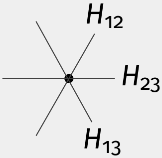
Example: In Type A, when $n = 3$:

$e_{H_{12}}$:

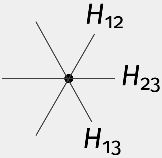


Multiplication is point-wise:

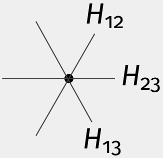
$e_{H_{12}}$:



$\cdot e_{H_{13}}$:



$= e_{H_{12}} e_{H_{13}}$:



Definition/Theorem (Varchenko-Gelfand, 1987).

The **associated graded Varchenko-Gelfand ring** $\mathcal{V}(\mathcal{A})$ has presentation

$$\mathbb{R}[e_{H_i} : H_i \in \mathcal{A}]/\mathcal{J}$$

where $\mathcal{J}$ is generated by:

1. **Idempotent relation**: $e_{H_i}^2$ for each $H_i \in \mathcal{A}$;
2. **Circuit relation**: For every circuit (e.g. minimal linear dependency) $C = (H_1, H_2, \cdots, H_m)$ in $\mathcal{A}$ such that $C = C^+ \sqcup C^-$,

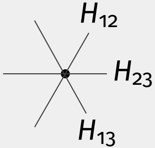
$$\sum_{i=1}^{m} c(i) e_{H_1} \cdots \widehat{e_{H_i}} \cdots e_{H_m}$$

where

$$c(i) = \begin{cases} 1 & \text{if } H_i \in C^-, \\ -1 & \text{if } H_i \in C^+. \end{cases}$$

Example: In Type A, when $n = 3$:

$\mathcal{A}_{S_3} =$



There is one circuit: $C = \{H_{12}, H_{23}, H_{13}\}$, which can be partitioned uniquely into $C^+ = H_{12}, H_{23}$ and $C^- = H_{13}$ so that

$$H_{12}^+ \cap H_{23}^+ \cap H_{13}^- = \emptyset.$$

Hence

$$\mathcal{V}(\mathcal{A}_{S_3}) = \mathbb{R}[e_{H_{12}}, e_{H_{23}}, e_{H_{13}}]/\Big\langle e_{H_{12}}^2, e_{H_{23}}^2, e_{H_{13}}^2, \\ e_{H_{12}}e_{H_{23}} - e_{H_{12}}e_{H_{13}} - e_{H_{23}}e_{H_{13}}\Big\rangle$$

Note: This matches Cohen's presentation of $H^* \operatorname{Conf}_3(\mathbb{R}^d)$, d odd

Claim:

$\mathcal{V}(\mathcal{A}_W)$ generalizes $H^* \operatorname{Conf}_n(\mathbb{R}^d)$ for odd $d \geq 3$...

Recall:

$$\operatorname{Conf}_n(\mathbb{R}^d) = \mathbb{R}^{dn} \setminus \Big(\bigcup_{1 \leq i < j \leq n} H_{ij} \otimes \mathbb{R}^d \Big)$$

Definition:

For any central hyperplane arrangement $\mathcal{A}$ of rank r,

$$\mathcal{M}^d(\mathcal{A}) := \mathbb{R}^{rd} \setminus \Big(\bigcup_{H_i \in \mathcal{A}} H_i \otimes \mathbb{R}^d \Big)$$

As in Type A, consider $H^* \mathcal{M}^d(\mathcal{A})$.

The cohomology of $\mathcal{M}^d(\mathcal{A}_W)$ depends on the **parity** of **d**!

Even case:

Theorem (Orlik-Solomon, 1980). When $d \geq 2$ is **even**, there is a W-equivariant ring isomorphism

$$H^* \mathcal{M}^d(\mathcal{A}_W) \cong_W \mathcal{OS}(\mathcal{A}_W),$$

where $\mathcal{OS}(\mathcal{A}_W)$ is the Orlik-Solomon algebra of $\mathcal{A}_W$.

Odd case:

Theorem (Moseley, 2017). When $d \geq 3$ is **odd**, there is a W-equivariant ring isomorphism

$$H^* \mathcal{M}^d(\mathcal{A}_W) \cong_W \mathcal{V}(\mathcal{A}_W),$$

where $\mathcal{V}(\mathcal{A}_W)$ is the associated-graded Varchenko-Gelfand ring of $\mathcal{A}_W$.

A comparison of the odd and even cases

	even $d \geq 2$	**odd $d \geq 3$**
$H^*\mathcal{M}^d(\mathcal{A}_W)$ is isomorphic to...	$\mathcal{OS}(\mathcal{A}_W)$	$\mathcal{V}(\mathcal{A}_W)$
multiplication in $H^*\mathcal{M}^d(\mathcal{A}_W)$	anti-commutative	commutative
presentation of $H^*\mathcal{M}^d(\mathcal{A}_W)$	$\mathbb{R}\langle e_H : H \in \mathcal{A}\rangle /$ idempotent & circuit relations	$\mathbb{R}[e_H : H \in \mathcal{A}] /$ idempotent & circuit relations
Special cases		
Cohen's presentation of $H^* \operatorname{Conf}_n(\mathbb{R}^d)$	$\mathcal{OS}(\mathcal{A}_{S_n})$	$\mathcal{V}(\mathcal{A}_{S_n})$
Xicotencatl's presentation of $H^* \operatorname{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^d)$	$\mathcal{OS}(\mathcal{A}_{B_n})$	$\mathcal{V}(\mathcal{A}_{B_n})$

Outline

Main Results

$$\text{Let } \beta_{W,k}(t) := \frac{\left(\frac{t+g-1}{g} - k\right)_k \cdot \left(\frac{t+1}{g}\right)_{r-k}}{\left(\frac{2}{g}\right)_r}.$$

Theorem (B—, 2020).
Let W be a real coincidental reflection group of rank r. For $0 \leq k \leq r$, the following are equivalent as W-representations:

1. $H^{k(d-1)}\mathcal{M}^d(\mathcal{A}_W)$ for odd $d \geq 3$,
2. $\mathcal{V}^k(\mathcal{A}_W)$, the k-th graded piece of the associated graded Varchenko-Gelfand ring
3. The σ_{r-k}-th eigenspace of the shuffle element $\mathcal{S}(W) \in \mathbb{R}W$
4. The representation $\mathbb{R}W\mathfrak{e}_{r-k}$ where $\mathfrak{e}_{r-k}$ is defined by

$$\sum_{k=0}^{r} t^k \mathfrak{e}_k = \sum_{w \in W} \beta_{W,\mathrm{des}(w)}(t) \cdot w.$$

Thank you for **listening**!

Future Directions

Complex Reflection Groups:

There are complex (non-real) coincidental reflection groups

These are precisely Shephard groups, which are the symmetry groups of complex polytopes

Question: To what extent does the story of the real Eulerian representations generalize to Shephard groups?

I would love to discuss any ideas in this direction!

Properties of the Eulerian representations

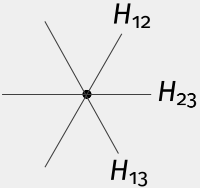
Many representation theoretic properties of $\mathfrak{e}_k$ in Type A are not known in other types!

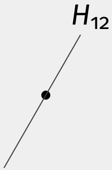
Currently: $\mathbb{R} S_n \mathfrak{e}_k$ has a “hidden” S_{n+1} action. I am working on generalizing this to type B using configuration spaces

What makes the coincidental groups special?

For $X \in \mathcal{L}(\mathcal{A})$, the restriction arrangement $\mathcal{A}^X$ is

$$\mathcal{A}^X := \{H \cap X : H \in \mathcal{A}, X \not\subset H\}.$$

$\mathcal{A} =$ 

$\mathcal{A}^{H_{12}} =$ 

Theorem: (Abramenko, 1994; Aguiar-Mahajan, 2017).

$\mathcal{A}^X$ is a reflection arrangement for every $X \in \mathcal{L}(\mathcal{A})$

if and only if

W is a (product of) coincidental reflection group(s)

When W is coincidental: $\mathcal{A}^X \cong \mathcal{A}^Y$ if and only if $\dim(X) = \dim(Y)$

Results for any Coxeter group

Let $[X] \in \mathcal{L}(\mathcal{A})/W$ be the W-orbit of $X \in \mathcal{L}(\mathcal{A})$.

Theorem (B–, 2020).

For any finite Coxeter group W and $[X] \in \mathcal{L}(\mathcal{A})/W$,

$$\underbrace{\mathbb{R}\,W\mathfrak{e}_{[X]}}_{\text{idempotent indexed by flat orbits}} \quad \cong_W \quad \underbrace{\mathcal{V}(\mathcal{A})_{[X]}}_{\text{decomposition of } \mathcal{V}(\mathcal{A}) \text{ by flat orbit}}$$

Proof techniques

Big idea:

Map $\mathcal{S}(W)$ into the Tits (face) semigroup algebra of $\mathcal{A}$

Relate eigenvalues of $\mathcal{S}(W)$ to restriction arrangements $\mathcal{A}^X$

Use the fact that when W is coincidental, $\mathcal{A}^X$ depends only on the dimension of X